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On the ERG approach in $3-d$ Well Developed Turbulence*

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Abstract

We apply the method of the Renormalization Group (GR), following the Polchinski point of view, to a model of well developed and isotropic fluid turbulence. The Galilei-invariance is preserved and a universality behavior, related to the change of the stochastic stirring force, is evident by the numerical results in the inertial region, where a scale-invariant behavior also appear. The expected power law of the energy spectrum ($q^{-\frac{5}{3}}$) is obtained and the Kolmogorov constant C_K agrees with experimental data.

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1. Introduction

The incompressible fluid turbulence, at very high Reynolds number ($\mathcal{R}$), is described by physical systems with a large number of degrees of freedom that are characterized by a substantial scale invariance. These peculiarities are investigated in Statistical Hydrodynamics by models in which the Navier-Stokes equations are supplemented with stochastic force terms and the analogies between scaling at high-Reynolds-number turbulence regime and equilibrium critical phenomena have been proposed. Attempts have been made to apply the powerful *renormalization-group* (RG) methods, which have proved so effective and illuminating in critical phenomena, to turbulence scaling [1],[2],[3]. De Dominicis and Martin [2] analysed these models near the critical dimension with a perturbative approach. Yakhot and Orszag [3] have extrapolated the perturbative RG study from the critical to physical dimension and, on this basis they predicted, in a long inertial range, various scaling laws, such as the Kolmogorov energy spectrum [4], including constant factors as the Kolmogorov constant, the Obukhov-Corrsin constant and the turbulent Prandtl number. But the numerical success of the Yakhot and Orszag's theory does not rest on firm theoretical grounds. Indeed the critical dimension ($d = 7$) is very far from the physical one ($d = 3$) and it is not evident that an ϵ expansion is the best procedure [5]. Another difficulty of the Yakhot and Orszag's method is related to the breaking of the Galilei invariance produced by the Wilson-Kadanoff RG analysis in the presence of an infinite number of marginal diagrams [5]. For these reasons the perturbative RG approach requires some ad hoc hypothesis [3] and produces only a phenomenological description of turbulence [5],[6],[7].

The goal of this paper is that of reconsidering the RG approach for the homogeneous isotropic turbulence, for an incompressible fluid, at very high Reynolds number, but in a non perturbative context. The point of view is very similar to that introduced by Polchinski [8], [9], [10] in the context of the Quantum Field Theory (QFT). The analogy with QFT follows from the description of our model in terms of a path-integral representation of the probability generating functionals, using a so-called Martin-Siggia-Rosen (MSR) action [11].

The dynamics of the model is characterized by the presence of two scales; one of them corresponds to the macroscopic size L of the system, and the mechanism which maintains the energy stationary is operating at this scale. The second, the Kolmogorov scale η_d, is an internal, typically smaller scale, at which the energy, after its transfer by a cascade process [7], is dissipated. When the dimensionless quantity $\frac{L}{\eta_d} = \mathcal{R}^{\frac{3}{4}}$ is very large the system admits a wide *inertial range*, where its statistical properties are universal, homogeneous, isotropic and self-similar. As in the Yakhot and Orzag's paper [3], the model describes true turbulence only in this long inertial range and the manifest scale invariance of this region is interpretable as the existence of a critical phase in the infinite volume limit.

The analysis of this paper is centered on the study of the flow equations for the effective action of an infrared regularized theory. This method involves an infinity of coupled differential equations and necessary requires an approximation scheme. The perturbative approximation [12], for the previous arguments, preserves an infinite number of terms and, in our case, no summability criterion of the perturbative series is known. In other schemes, as the derivative expansion approximation [13], [14] the correlations functions are strongly dependent on the particular form of the stirring force also in the long inertial region. The problem of the approximation has its solution in the fact that, in the momentum space, the size of the spectrum of two point stirring force correlation function is of order $O\left(\frac{1}{L}\right)$. From this property, using the same flow equation, it is possible to see that, in the inertial range (and only in this region), some proper correlation functions are smaller then others.

The paper is organized as follow:

In Section 2 we summarize the model concerning the Navier-Stokes equation with a stochastic force term following the Martin, Siggia and Rose point of view.

In Section 3 we analyze the Galilei-invariance of the model and we write the Ward Identities for the functional generators.

Section 4 is devoted to the equation flow referred to the scale parameters which characterize the model; we also analyze the boundary conditions and some properties of the vertex functions are discussed in Appendix A.

Section 5 contains a discussion concerning the approximation scheme with the relevant hypotheses and a plausibility argument which further considered in Appendix B.

Some numerical results, concerning the renormalized viscosity and the Kolmogorov constant, are contained in Section 6.

Finally some conclusive remarks are contained in Section 7.

2. The model

Let us consider the Navier-Stokes equations, supplemented with a stochastic force term,

$$\frac{\partial}{\partial t}u^{\alpha}(t,\vec{x}) + u^{\beta}(t,\vec{x})\frac{\partial}{\partial x^{\beta}}u^{\alpha}(t,\vec{x}) = -\frac{1}{\rho}\frac{\partial}{\partial x_{\alpha}}p(t,\vec{x}) + \nu \bigtriangledown^2 u^{\alpha}(t,\vec{x}) + f^{\alpha}(t,\vec{x}) \qquad (2.1)$$

and with the incompressibility constraint

$$\frac{\partial}{\partial x^{\alpha}} u^{\alpha}(t,\vec{x}) = 0. \tag{2.2}$$

The fields u^{α} are the three components of the velocity field and p is the pressure field; the effects of the boundary conditions over the zero mean value field u^{α} are summarized by the stochastic source f^{α} which is a Gaussian random force with mean zero and covariance

$$\langle f^{\alpha}(t,\vec{x}) f^{\beta}(t',\vec{y})\rangle = F^{\alpha\beta}\left(t,t';\vec{x},\vec{y}\right) = 2P^{\alpha\beta}(\vec{\bigtriangledown}_x) N(\vec{x}-\vec{y})\delta(t-t'). \tag{2.3}$$

$P^{\alpha\beta}(\vec{\bigtriangledown}_x)$ is the projection onto solenoidal vector fields, required to maintain the incompressibility constraint (2.2). Since the *stirring force* f^{α} should mimic the instabilities occurring near the boundaries, its correlation length must be of the same order of the system size. In other words if L is the length corresponding to the size of the system and

$$\langle f^{\alpha}(q^0,\vec{q}) f^{\beta}(q^{0\prime},\vec{q}')\rangle = F^{\alpha\beta}(q^0,q^{0\prime};\vec{q},\vec{q}') = 2(2\pi)^4 P^{\alpha\beta}(q)h(q)\delta^3(\vec{q}+\vec{q}')\delta(q^0+q^{0\prime}). \tag{2.4}$$

is the Fourier transform of (2.3), the function $h(q)$ must vanish for $q > \frac{1}{L} = m$. The explicit form of this function is not very important if the self-similarity behaviour is realized, but in the stationary limit the energy dissipated by the viscous forces must be compensated by the energy introduced through the boundaries and a physical normalization condition is needed, e. g.

$$\langle \mathcal{E} \rangle = 2\int \frac{d^3q}{(2\pi)^3} h(q), \tag{2.5}$$

where $\mathcal{E}$ is the rate of power dissipated by a unit mass of fluid and can be calculated by the local quantity [7],[15]

$$\mathcal{E}(t,\vec{x}) = \frac{1}{2}\nu\left(\partial_{\alpha}u_{\beta}(t,\vec{x}) + \partial_{\beta}u_{\alpha}(t,\vec{x})\right)^2 .$$

The probability generating functional is given by the functional integral

$$\mathcal{W}\left(J,\hat{J},p,\hat{p}\right) = \int \mathcal{D}u\mathcal{D}\hat{u}\mathcal{D}f e^{i\int d\hat{x}\mathcal{L}(u,\hat{u},p,\hat{p},f,J,\hat{J})} e^{-\frac{1}{2}\int d\hat{x}d\hat{y} f^{\alpha}(\hat{x})F^{-1}_{\alpha\beta}(\hat{x},\hat{y})f^{\beta}(\hat{y})} \tag{2.6}$$

where we have set $\hat{x} \equiv (t,\vec{x})$. $\mathcal{L}$ is the Navier-Stokes density of Lagrangian

$$\mathcal{L}(\hat{x}) = \hat{u}^{\alpha}(\hat{x})\left(\frac{\partial}{\partial t} - \nu\bigtriangledown^2\right)u_{\alpha}(\hat{x}) + \hat{u}^{\alpha}(\hat{x})u_{\beta}(\hat{x})\frac{\partial}{\partial x^{\beta}}u_{\alpha}(\hat{x})$$

$$+\frac{1}{\rho}\hat{u}^\alpha(\hat{x})\frac{\partial}{\partial x^\alpha}p(\hat{x})+\frac{1}{\rho}\hat{p}(\hat{x})\frac{\partial}{\partial x^\alpha}u^\alpha(\hat{x})+\hat{J}^\alpha(\hat{x})u_\alpha(\hat{x})+\hat{u}^\alpha(\hat{x})J_\alpha(\hat{x}).$$

The field $\hat{u}^\alpha$ is the conjugate variable to the velocity field u^α, p (the pressure field) and its conjugate variable $\hat{p}$ are the two Lagrangian's multipliers related to the solenoidal constraint for the fields u^α and $\hat{u}^\alpha$ and finally J^α and $\hat{J}^\alpha$ are external sources. In (2.6) it is possible to integrate over the functional gaussian measure $\mathcal{D}fe^{-\frac{1}{2}\int f^\alpha F^{-1}_{\alpha\beta}f^\beta}$ and to reduce the space of the solutions only to the independent transverse degrees of freedom. We introduce the transverse variables $v^\alpha = P^{\alpha\beta}(\bigtriangledown)u^\beta$, $\hat{v}^\alpha = P^{\alpha\beta}(\bigtriangledown)\hat{u}^\beta$ and, after the integration over f^α, we obtain [6] the Martin, Siggia and Rose probability generating functional [11]

$$\mathcal{W}(J,\hat{J})=\int\mathcal{D}v\mathcal{D}\hat{v}e^{i\left(S(v,\hat{v})+\int d\hat{x}\hat{J}^\alpha(\hat{x})v_\alpha(\hat{x})+\int d\hat{x}\hat{v}^\alpha(\hat{x})J_\alpha(\hat{x})\right)} \tag{2.7}$$

where the action $S(v,\hat{v})$ is

$$\begin{aligned} S(v,\hat{v}) = \textstyle\int dt \int d^3x \hat{v}^\alpha(t,\vec{x}) \left[\left(\partial_t-\nu\bigtriangledown^2\right)v_\alpha(t,\vec{x})+v^\beta(t,\vec{x})\partial_\beta v^\alpha(t,\vec{x})\right] \\ +\tfrac{i}{2}\textstyle\int dt\int d^3x\int d^3y\hat{v}^\alpha(t,\vec{x})N_{\alpha\beta}(\vec{x}-\vec{y})\hat{v}^\beta(t,\vec{y}). \end{aligned} \tag{2.8}$$

The multitime and multipoint correlation functions are obtained by functional derivatives of (2.7) with respect to the external sources $\hat{J}^\alpha$ and J^α. The connected functional is defined as

$$ln\mathcal{W}(J,\hat{J})=i\mathcal{Z}(J,\hat{J}) \tag{2.9}$$

and the generic connected correlation function is given by

$$\langle v^{\alpha_1}(\hat{x}_1)...v^{\alpha_n}(\hat{x}_n)\hat{v}^{\alpha_{n+1}}(\hat{x}_{n+1})...\hat{v}^{\alpha_{n+m}}(\hat{x}_{n+m})\rangle_c$$

$$=(-i)^{n+m}e^{-i\mathcal{Z}(J,\hat{J})}\frac{\delta}{\delta\hat{J}^{\alpha_1}(\hat{x}_1)}...\frac{\delta}{\delta\hat{J}^{\alpha_n}(\hat{x}_n)}\frac{\delta}{\delta J^{\alpha_{n+1}}(\hat{x}_{n+1})}...\frac{\delta}{\delta J^{\alpha_{n+m}}(\hat{x}_{n+m})}e^{i\mathcal{Z}(J,\hat{J})}\Big|_{\hat{J}=J=0} \tag{2.10}$$

where the stability conditions

$$\frac{\delta\mathcal{Z}(J,\hat{J})}{\delta\hat{J}^\alpha(\hat{x})}\Big|_{\hat{J}=J=0}=\frac{\delta\mathcal{Z}(J,\hat{J})}{\delta J^\alpha(\hat{x})}\Big|_{\hat{J}=J=0}=0 \tag{2.11}$$

are satisfied. Finally the convex vertex functional (the effective action) $\Gamma(v,\hat{v})$ is defined by the Legendre transformation [6]

$$\mathcal{Z}(J,\hat{J})=\Gamma(v,\hat{v})+\int d\hat{x}\left(\hat{J}^\alpha(\hat{x})v_\alpha(\hat{x})+\hat{v}^\alpha(\hat{x})J_\alpha(\hat{x})\right). \tag{2.12}$$

Since from (2.12) we have the relation

$$\frac{\delta\Gamma(v,\hat{v})}{\delta v^\alpha(\hat{x})} = -\hat{J}^\alpha(\hat{x}), \quad \frac{\delta\Gamma(v,\hat{v})}{\delta \hat{v}^\alpha(\hat{x})} = -J^\alpha(\hat{x}),$$

the stability conditions (2.11) are rewritten

$$\left.\frac{\delta\Gamma(v,\hat{v})}{\delta v^\alpha(\hat{x})}\right|_{u=\hat{u}=0} = \left.\frac{\delta\Gamma(v,\hat{v})}{\delta \hat{v}^\alpha(\hat{x})}\right|_{u=\hat{u}=0} = 0. \tag{2.13}$$

We conclude this section observing that the knowledge of the functional $\mathcal{W}(J,\hat{J})$, and then of $\mathcal{Z}(J,\hat{J})$ and $\Gamma(v,\hat{v})$, gives a complete solution of the model in terms of correlation functions. The direct calculation of the functional generator from the Navier-Stokes equation (2.1), with suitable asymptotic boundary conditions, is a formidable task which may be simplified studying the constraints deriving from the symmetries of the model and the scaling properties connected to the variation of the dimensional parameters. The following sections are devoted to developing this point of view.

3. Galilei invariance and Ward's identities

In the theory of turbulence we consider the very important hypothesis that, in the limit of infinite Reynolds number, all the symmetries of the Navier-Stokes equation, that are broken by the mechanism producing the turbulent flow, are restored in a statistical sense at small scale and away from the boundaries [16]. In our model, the last term in eq.(2.8) is Galilei invariant, as the original Navier-Stokes equation, and this symmetry will be transferred to all correlation functions. Then, as a physical system is described by its symmetries, if scale invariance is also recovered in the inertial range region, the model will provide a good description of the isotropic turbulence in this region. In this section we study the constraint deriving from Galilei-invariance.

Let us consider the following coordinates transformations

$$t \to t' = t, \quad x^\alpha \to x'^\alpha = x^\alpha + c^\alpha t$$

with corresponding field transformations

$$v^\alpha(t,\vec{x}) \to v'^\alpha(t,\vec{x}') = v^\alpha(t,\vec{x}) - c^\alpha,$$

$$\hat{v}^\alpha(t,\vec{x}) \to \hat{v}'^\alpha(t,\vec{x}') = \hat{v}^\alpha(t,\vec{x}),$$

where c^α is a constant velocity. We have then the Lie's derivatives

$$\delta v^\alpha(t,\vec{x}) = -c^\lambda t \partial_\lambda v^\alpha(t,\vec{x}) - c^\alpha,$$
$$\delta \hat{v}^\alpha(t,\vec{x}) = -c^\lambda t \partial_\lambda \hat{v}^\alpha(t,\vec{x}). \tag{3.1}$$

From (3.1) we obtain the following Ward identity for the functional generator $\mathcal{W}(J,\hat{J})$

$$\mathcal{G}^\lambda \mathcal{W}(J,\hat{J}) \equiv \int d\hat{x} \left(it\hat{J}^\alpha(\hat{x}) \frac{\partial}{\partial x^\lambda} \frac{\delta}{\delta \hat{J}^\alpha(\hat{x})} + itJ^\alpha(\hat{x}) \frac{\partial}{\partial x^\lambda} \frac{\delta}{\delta J^\alpha(\hat{x})} - \hat{J}^\lambda(\hat{x}) \right) \mathcal{W}(J,\hat{J}) = 0. \tag{3.2}$$

The functional-differential operator $\mathcal{G}^\lambda$ carries a representation of the infinitesimal Galilei transformations and it is a generator of a closed Lie algebra. Indeed the model is also invariant under time and space translations, given by the Ward identities

$$\mathcal{H}\mathcal{W}(J,\hat{J}) \equiv \int d\hat{x} \left(i\hat{J}^\alpha(\hat{x}) \frac{\partial}{\partial t} \frac{\delta}{\delta \hat{J}^\alpha(\hat{x})} + iJ^\alpha(\hat{x}) \frac{\partial}{\partial t} \frac{\delta}{\delta J^\alpha(\hat{x})} \right) \mathcal{W}(J,\hat{J}) = 0, \tag{3.3}$$

$$\mathcal{P}^\lambda \mathcal{W}(J,\hat{J}) \equiv \int d\hat{x} \left(i\hat{J}^\alpha(\hat{x}) \frac{\partial}{\partial x^\lambda} \frac{\delta}{\delta \hat{J}^\alpha(\hat{x})} + iJ^\alpha(\hat{x}) \frac{\partial}{\partial x^\lambda} \frac{\delta}{\delta J^\alpha(\hat{x})} \right) \mathcal{W}(J,\hat{J}) = 0 \tag{3.4}$$

which obey the commutation relations

$$\left[\mathcal{G}^\lambda, \mathcal{H}\right] = -\mathcal{P}^\lambda, \quad \left[\mathcal{G}^\lambda, \mathcal{G}^\gamma\right] = \left[\mathcal{G}^\lambda, \mathcal{P}^\gamma\right] = \left[\mathcal{P}^\lambda, \mathcal{P}^\gamma\right] = \left[\mathcal{P}^\lambda, \mathcal{H}\right] = 0. \tag{3.5}$$

Analogous relations hold for the functionals $\mathcal{Z}(J,\hat{J})$ and $\Gamma(v,\hat{v})$. Introducing the Fourier transform

$$\tilde{v}^\alpha(\hat{q}) = \frac{1}{(2\pi)^2} \int dq^0 \int d^3q e^{-iq^0t - iq_\lambda x^\lambda} v^\alpha(\hat{x}), ... \tag{3.6}$$

the previous relations are rewritten, for example in terms of the vertex functional, as

$$-\mathcal{G}^\lambda \Gamma(v,\hat{v}) = \int d\hat{q} \left(q^\lambda \frac{\partial}{\partial q^0} \tilde{v}^\alpha(\hat{q}) \frac{\delta}{\delta \tilde{v}^\alpha(\hat{q})} + q^\lambda \frac{\partial}{\partial q^0} \tilde{\hat{v}}^\alpha(\hat{q}) \frac{\delta}{\delta \tilde{\hat{v}}^\alpha(\hat{q})} \right) \Gamma(v,\hat{v}) + \frac{\delta}{\delta \tilde{v}^\lambda(\hat{0})} \Gamma(v,\hat{v}) = 0, \tag{3.7}$$

$$-\mathcal{H}\Gamma(v,\hat{v}) = \int d\hat{q} \left(q^0 \tilde{v}^\alpha(\hat{q}) \frac{\delta}{\delta \tilde{v}^\alpha(\hat{q})} + q^0 \tilde{\hat{v}}^\alpha(\hat{q}) \frac{\delta}{\delta \tilde{\hat{v}}^\alpha(\hat{q})} \right) \Gamma(v,\hat{v}) = 0, \tag{3.8}$$

$$-\mathcal{P}^\lambda \Gamma(v,\hat{v}) = \int d\hat{q} \left(q^\lambda \tilde{v}^\alpha(\hat{q}) \frac{\delta}{\delta \tilde{v}^\alpha(\hat{q})} + q^\lambda \tilde{\hat{v}}^\alpha(\hat{q}) \frac{\delta}{\delta \tilde{\hat{v}}^\alpha(\hat{q})} \right) \Gamma(v,\hat{v}) = 0. \tag{3.9}$$

In order to derive from the previous identities, some general properties of correlation functions, we introduce the following simplified notation for the n-points vertex functions.

$$\left.\frac{\delta^{n+m}\Gamma(v,\hat{v})}{\delta\tilde{v}^{\alpha_1}(\hat{q}_1)..\delta\tilde{v}^{\alpha_n}(\hat{q}_n)\delta\tilde{\hat{v}}^{\alpha_{n+1}}(\hat{q}_{n+1})..\delta\hat{v}^{\alpha_{n+m}}(0)}\right|_{v,\hat{v}=0} = \Gamma_{\tilde{v}^{\alpha_1}(\hat{q}_1)..\tilde{v}^{\alpha_n}(\hat{q}_n)\tilde{\hat{v}}^{\alpha_{n+1}}(\hat{q}_{n+1})..\hat{v}^{\alpha_{n+m}}(0)}. \quad (3.10)$$

We note that the last field in (3.10) is written in configuration space, consequently only the independent momenta appear and no kinematic δ singularity is present.

The first non trivial identity is

$$k^\lambda \frac{\partial}{\partial k^0}\Gamma_{\tilde{v}^\alpha(\hat{k})\hat{v}^\beta(0)} + \Gamma_{\tilde{v}^\lambda(\hat{0})\tilde{v}^\alpha(\hat{k})\hat{v}^\beta(0)} = 0, \quad (3.11)$$

and a consequence of this relation is, for example, that the coefficients of the two terms of the convective derivative $\partial_t + v^\alpha\partial_\alpha$ have the same value. Indeed from power counting and tensorial analysis we set

$$\Gamma_{\tilde{v}^\alpha(\hat{k})\hat{v}^\beta(0)} = \left(-i\sigma_0 k^0 + \sigma_\nu k^2 + \Pi(\hat{k})\right)\delta^{\alpha\beta},$$

and

$$\Gamma_{\tilde{v}^\lambda(\hat{p})\tilde{v}^\alpha(\hat{k})\hat{v}^\beta(0)} = i\left(\delta^{\beta\lambda}p^\alpha + \delta^{\beta\alpha}k^\lambda\right)\left(\sigma_{\mathcal{R}} + \Sigma(\hat{p},\hat{k})\right)$$

with $\Pi(\hat{0}) = \Sigma(\hat{0},\hat{0}) = 0$. From (3.11) we obtain the constraints

$$\sigma_0 = \sigma_{\mathcal{R}} \quad and \quad \frac{\partial}{\partial k^0}\Pi(\hat{k}) = i\Sigma(\hat{0},\hat{k}). \quad (3.12)$$

Analogous relations follow from the general Ward identity of the proper vertex correlation function that can be written as

$$k^\lambda\frac{\partial}{\partial k^0}\Gamma_{\tilde{v}^\alpha(\hat{k})\tilde{v}^\gamma(\hat{p})...\hat{v}^\beta(0)} + p^\lambda\frac{\partial}{\partial p^0}\Gamma_{\tilde{v}^\alpha(\hat{k})\tilde{v}^\gamma(\hat{p})...\hat{v}^\beta(0)} + ... + \Gamma_{\tilde{v}^\lambda(\hat{0})\tilde{v}^\alpha(\hat{k})\tilde{v}^\gamma(\hat{p})...\hat{v}^\beta(0)} = 0. \quad (3.13)$$

4. The evolution equation

The mechanism of energy input in the system, necessary to preserve the stationary regime, is described by the last term of the Martin, Siggia and Rose action (2.8) and it contains a

characteristic physical scale. As we have already observed this scale is given by the parameter m that is the width, in momentum space, of the two points stirring force correlation function support and $m = \frac{1}{L}$ where L is of the order of the linear dimension of the system. In a real physical system an other short-distance scale, related to the dissipative region, is present; the relevant parameter is the Kolmogorov's cutoff $K_d = \frac{1}{\eta_d}$. The two scales m and K_d are very important in the physical description of the system. In fact the dimensionless quantity

$$\left(\frac{K_d}{m}\right)^{\frac{4}{3}} = \mathcal{R} \tag{4.1}$$

is the Reynolds number [7] which is a measure of the nonlinearity occurring at the scale L. If the Reynolds number is very high the system develops a wide domain of the Fourier space (*the inertial range*) in which the non linear terms dominate. If we consider the momenta in a region included in this domain, such that $m \ll q \ll K_d$, both boundary and viscous terms can be neglected and we expect the system to be strongly self similar, homogeneous in space and isotropic. This behavior of the system is interpretable as the presence of a critical phase in the limit $\mathcal{R} \to \infty$. In order to reproduce the previous phenomenology starting from the model described by the action (2.8), we consider the Kolmogorov cutoff K_d as a reference scale and we study the system assuming the other scale m to be very very small compared with K_d. In other words we study the system near the infrared fixed point (if this fixed point exists) connected to the previous described critical phase. We expect that, for momenta q such that $m \ll q \ll K_d$, the self similar behaviour becomes manifest and that correlation functions obey, with good approximation, the behaviour given by dimensional analysis. In particular the energy spectrum $E(q)$ must follow a power law of the form [4]

$$E(q) = C_K \mathcal{E}^{\frac{2}{3}} q^{-\frac{5}{3}}, \tag{4.2}$$

where C_K is a constant of the order of unity and the renormalization group methods allow a quantitative estimation of the C_K Kolmogorov constant. Experimental tests prove that the scaling behavior of correlation functions of order higher then 3 has some corrections. Such corrections are generally linked to the intermittent behavior of turbulent systems [16], i. e. to strong, nonlinear, rare events that are usually associated with the instantonic configurations. This aspect is not discussed in the approach here analysed.

In order to obtain an evolution equation for the model (2.8) we introduce the six components field $\tilde{\Psi}^a(\hat{q}) \equiv \left(\tilde{v}^\alpha(\hat{q}), \tilde{\tilde{v}}^\alpha(\hat{q})\right)$ and the density of lagrangian

$$\mathcal{L}(q;m) = \mathcal{L}^0(q) + \frac{1}{2}\tilde{\Psi}^a(-\hat{q}) R_{ab}(q;m) \tilde{\Psi}^b(\hat{q}) \tag{4.3}$$

where $\mathcal{L}^0(q)$ is the original Navier-Stokes lagrangian (independent of the scale m) without stirring force, which is totally included in the last term where the matrix $R_{ab}(q;m)$ is given by

$$R_{ab}(q;m) = \begin{pmatrix} 0 & 0 \\ 0 & 2ih(q;m)P^{\alpha\beta}(q) \end{pmatrix}. \tag{4.4}$$

Concerning the function $h(q;m)$, it must satisfy the physical constraint (2.5) and its momentum spectrum should be of width m. A possible choice is

$$h(q;m) = \frac{D_0}{m^3}\chi\left(\frac{q}{m}\right), \tag{4.5}$$

where, for example

$$\chi\left(\frac{q}{m}\right) = \xi\left(\frac{q}{m}\right)e^{-\frac{q^2}{m^2}}.$$

D_0 is a dimensional parameter fixed by the constraint (2.5), once the homogeneous function $\chi\left(\frac{q}{m}\right)$ is specified. The function (4.5) is very similar to that chosen by P. Brax [17] and as the parameter m is different from zero, the density of lagrangian (4.3) corresponds to a model with a Galilei-invariant infrared regularization. The ultraviolet convergence is assured by the same function (4.5) and by the existence of a physical cutoff $\Lambda_0 > K_d$ where, for momenta $q \sim \Lambda_0$, all statistical fluctuations rapidly vanish due to the dissipative effects.

Taking into account that all the m dependence of the functional $\Gamma(\Psi;m)$ ($\Gamma(\Psi;m) \equiv \Gamma(v,\hat{v})$) is contained in the function $h(q;m)$, one easily [8], [12], [18] derives an evolution equation in m

$$m\partial_m\left(\Gamma(\Psi;m) - \frac{1}{2}\int d\hat{q}\tilde{\Psi}^a(-\hat{q})R_{ab}(q;m)\tilde{\Psi}^b(\hat{q})\right)$$

$$= -\frac{i}{2}\int\frac{d\hat{q}}{(2\pi)^4}m\partial_m R_{ab}(q;m)\Delta_{\Psi_b\Psi_c}(\hat{q};m)\frac{\delta^2\bar{\Gamma}(\Psi;m)}{\delta\tilde{\Psi}_c(\hat{q})\delta\tilde{\Psi}_l(-\hat{q})}\Delta_{\Psi_l\Psi_a}(\hat{q};m), \tag{4.6}$$

where $\Delta_{\Psi_b\Psi_c}(\hat{q};m)$ is the full propagator which, in term of the two points proper correlation functions, is given by the matrix

$$\Delta_{\Psi_a\Psi_b}(\hat{q};m) = \Gamma^{-1}_{\tilde{\Psi}_a(\hat{q})\Psi_b(0)} = \begin{pmatrix} -\frac{\Gamma_{\tilde{\hat{v}}^\gamma(\hat{q})\hat{v}^\rho(0)}P^{\rho\sigma}(q)}{\Gamma_{\tilde{v}^\alpha(\hat{q})\hat{v}^\gamma(0)}\Gamma_{\tilde{\hat{v}}^\sigma(\hat{q})v^\beta(0)}} & \frac{P^{\alpha\sigma}(q)}{\Gamma_{\tilde{v}^\sigma(\hat{q})\hat{v}^\beta(0)}} \\ \frac{P^{\alpha\sigma}(q)}{\Gamma_{\tilde{\hat{v}}^\sigma(\hat{q})v^\beta(0)}} & 0 \end{pmatrix}. \tag{4.7}$$

The auxiliary functional $\frac{\delta^2\bar{\Gamma}(\Psi;m)}{\delta\tilde{\Psi}_a(\hat{q})\delta\tilde{\Psi}_b(\hat{q}')} \equiv \bar{\Gamma}_{ab}(\hat{q},\hat{q}';m)$ is defined by the integral equation

$$\bar{\Gamma}_{ab}(\hat{q},\hat{q}';m) = \Gamma^{int}_{ab}(\hat{q},\hat{q}';m) - \int d\hat{q}''\bar{\Gamma}_{cb}(-\hat{q}'',\hat{q}';m)\Delta_{\Psi_d\Psi_c}(\hat{q}'';m)\Gamma^{int}_{ad}(\hat{q},\hat{q}'';m) \tag{4.8}$$

where $\Gamma^{int}_{ab}(\hat{q},\hat{q}';m)$ is obtained by the functional

$$\Gamma^{int}\left(\Psi;m\right)=\Gamma\left(\Psi;m\right)-\frac{1}{2}\int d\hat{q}\tilde{\Psi}_a(-\hat{q})\Gamma_{\tilde{\Psi}_a(\hat{q})\Psi_b(0)}\tilde{\Psi}_b(-\hat{q}), \tag{4.9}$$

i, e. the generator $\Gamma\left(\Psi;m\right)$ without the bilinear terms. In eq.(4.7) we have assumed that $\Gamma_{v^\alpha v^\beta}=0$; in perturbation theory all vertex functions containing only v^α field vanish, but this property is an exact consequence of (4.6) and the boundary conditions (see appendix A). From (4.6) we obtain the equation for the vertex correlation functions. In order to tackle specific problems, consider the flow equations for the two points vertex functions $\Gamma_{\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}$ and $\Gamma_{\tilde{\hat{v}}^\alpha(\hat{p})\hat{v}^\beta(0)}$; they are

$$\begin{aligned}m\partial_m\Gamma_{\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}&=\int\frac{d\hat{q}}{(2\pi)^4}m\partial_m h(q;m)\Big(\Delta_{\hat{v}^\gamma v^\rho}(\hat{q};m)\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\delta(-\hat{q})\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}\Delta_{v^\delta\hat{v}^\gamma}(\hat{q};m)\\&-\Delta_{\hat{v}^\gamma v^\rho}(\hat{q};m)\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\alpha(\hat{p})\Psi^s(0)}\Delta_{\Psi^s\Psi^r}(\hat{q}+\hat{p};m)\Gamma_{\tilde{\Psi}^r(\hat{q}+\hat{p})\tilde{v}^\delta(-\hat{q})\hat{v}^\beta(0)}\Delta_{v^\delta\hat{v}^\gamma}(\hat{q};m)\\&-\Delta_{\hat{v}^\gamma v^\rho}(\hat{q};m)\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{\Psi}^s(\hat{p}-\hat{q})\hat{v}^\beta(0)}\Delta_{\Psi^s\Psi^r}(\hat{p}-\hat{q};m)\Gamma_{\tilde{\Psi}^r(\hat{q}-\hat{p})\tilde{v}^\alpha(\hat{p})v^\delta(0)}\Delta_{v^\delta\hat{v}^\gamma}(\hat{q};m)\Big)\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}&m\partial_m\left(\Gamma_{\tilde{\hat{v}}^\alpha(\hat{p})\hat{v}^\beta(0)}-2ih(p;m)P^{\alpha\beta}(p)\right)\\&=\int\frac{d\hat{q}}{(2\pi)^4}m\partial_m h(q;m)\Big(\Delta_{\hat{v}^\gamma v^\rho}(\hat{q};m)\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\delta(-\hat{q})\tilde{\hat{v}}^\alpha(\hat{p})\hat{v}^\beta(0)}\Delta_{v^\delta\hat{v}^\gamma}(\hat{q};m)\\&-\Delta_{\hat{v}^\gamma v^\rho}(\hat{q};m)\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{\hat{v}}^\alpha(\hat{p})\Psi^s(0)}\Delta_{\Psi^s\Psi^r}(\hat{q}+\hat{p};m)\Gamma_{\tilde{\Psi}^r(\hat{q}+\hat{p})\tilde{v}^\delta(-\hat{q})\hat{v}^\beta(0)}\Delta_{v^\delta\hat{v}^\gamma}(\hat{q};m)\Big).\end{aligned} \tag{4.11}$$

In the r. h. s. of the previous equations there appear three and four points vertex functions that are defined by flow equations involving higher correlations thus producing an infinite system. We will analyze the fundamental point of truncating this system in another section, studying here only the general properties and the boundary conditions.

If there is an infrared fixed point in the limit $m\to 0$, it is convenient to study the system near this point. For this reason we measure all momenta in terms of the spectral parameter m and set

$$p^0=\tau m^2,\quad |\vec{p}|=p=xm,\quad q^0=\eta m^2,\quad |\vec{q}|=q=ym,... \tag{4.12}$$

We suppose that the spectral parameter m is very small compared with the Kolmogorov cutoff K_d. The space of the x-like variables is then divided into three regions. In the first region (the infrared region) the p momenta are smaller then m and

$$x<1.$$

Here all correlation functions are strongly dependent on the form of the stirring force. In the second region (the inertial region) the p momenta are finite and m is very small compared with these momenta. Here we have

$$x \gg 1.$$

In this region we expect a strong self-similarity behaviour of the system, provided that

$$x \ll \frac{K_d}{m}.$$

The last region is the dissipative region, where

$$x > \frac{K_d}{m}$$

The boundary conditions for eq.s(4.10) and (4.11), and for all possible flow equation, are obtained in this region by the consideration that for

$$x = x_0 = \frac{\Lambda_0}{m} \sim \infty, \quad where \quad \Lambda_0 > K_d$$

all statistical fluctuations vanish as a consequence of the dissipative effects. In other words for any external variable x of the order $\frac{\Lambda_0}{m} \sim \infty$ all correlation vertex functions with a field content different from the vertices of the original Navier-Stokes action, without the stirring force term, vanish. Therefore the boundary value of the effective action is the classical Navier-Stokes action.

From equations (4.10), (4.11) and the previous considerations it is possible to extract some general informations about the vertex correlation functions. Let us consider the two point vertex functions $\Gamma_{\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}$, that we rewrite, in terms of the rescaled variables and taking into account the power counting analysis, as

$$\Gamma_{\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)} = \Gamma^m_{\tilde{v}^\alpha(\tau,x)\hat{v}^\beta(0)} = m^2 \left[-il\left(\tau, \frac{D_0^{\frac{1}{2}}}{m^2}\right)\tau + f\left(\tau, x, \frac{D_0^{\frac{1}{2}}}{m^2}\right)x^2 \right] P^{\alpha\beta}(p), \tag{4.13}$$

$$\Gamma_{\tilde{\hat{v}}^\alpha(\hat{p})\hat{v}^\beta(0)} = \Gamma^m_{\tilde{\hat{v}}^\alpha(\tau,x)\hat{v}^\beta(0)} = 2i\frac{D_0}{m^3}\left[\chi(x) + M\left(\tau, x, \frac{D_0^{\frac{1}{2}}}{m^2}\right)\right] P^{\alpha\beta}(p). \tag{4.14}$$

These functions are defined, at $m \neq 0$, for all values of the variable τ and x. In particular they satisfy $\Gamma^m_{\tilde{v}^\alpha(0,0)\hat{v}^\beta(0)} = 0$ and $M(\tau, 0, ..) = 0$ for any $m \neq 0$. It is possible also to see that all vertex functions (with the exclusion of $\Gamma_{\hat{v}\hat{v}}$) are almost proportional to the momenta of the $\hat{v}$ fields. If we go back to equation (4.11) we see that some inconsistencies are present: indeed the r. h. s. of this equation, for $x = 0$ is different from zero due to infrared divergences of the full propagator.

This apparent inconsistency is caused by the fact that in the function $h(x;m) = \frac{D_0}{m^3}\chi(x)$ the value $x=0$ is not accessible due to the finite size of the system and therefore we must define

$$\chi(0) = 0. \tag{4.15}$$

The equations (4.11), (4.10) are defined for arbitrary values of the variable x and the previous condition for the function $h(x;m)$ at $x=0$ is a physical regularization connected with the infinite volume limit. Taking into account condition (4.15) we obtain a flow equation for the quantity $\Gamma^m_{\tilde{v}^\alpha(\tau,0)\hat{v}^\beta(0)} = -im^2 l(\tau,..)\tau$

$$(m\partial_m - 2\tau\partial_\tau)\left[m^2 l\left(\tau, \frac{D_0^{\frac{1}{2}}}{m^2}\right)\tau\right] = 0 \to (m\partial_m - 2\tau\partial_\tau)\, l\left(\tau, \frac{D_0^{\frac{1}{2}}}{m^2}\right) = 0$$

which has the general integral

$$l = l\left(D_0^{-\frac{1}{2}} m^2 \tau\right).$$

But the asymptotic boundary conditions in $x = \frac{\Lambda_0}{m}$ (and $\tau = \frac{\Lambda_0^2}{m^2}$) enforce $l\left(D_0^{-\frac{1}{2}} m^2 \tau\right) = 1$, independently of the values of the m variable. Therefore we have the exact result

$$l(\tau,..) \equiv 1.$$

Thus the two points vertex function is rewritten as

$$\Gamma^m_{\tilde{v}^\alpha(\tau,x)\hat{v}^\beta(0)} = m^2\left[-i\tau + f\left(\tau, x, \frac{D_0^{\frac{1}{2}}}{m^2}\right)x^2\right]P^{\alpha\beta}(p). \tag{4.16}$$

From the Ward identity (3.11) and from (3.12) we have another exact result concerning the three points vertex function $\Gamma_{\tilde{v}^\lambda(\hat{q})\tilde{v}^\alpha(\hat{p}-\hat{q})\hat{v}^\beta(0)}$ which is

$$\begin{aligned}\Gamma_{\tilde{v}^\lambda(\hat{q})\tilde{v}^\alpha(\hat{p}-\hat{q})\hat{v}^\beta(0)} &= \Gamma^m_{\tilde{v}^\lambda(\eta,y)\tilde{v}^\alpha(\tau-\eta,x-y)\hat{v}^\beta(0)} \\ &= imx\left(\delta^{\beta\lambda}n^\alpha + \delta^{\beta\alpha}n^\lambda\right)(1+\Sigma(\tau,\eta,x,y,..))\end{aligned} \tag{4.17}$$

where $p^\alpha = mxn^\alpha$. These exact results, which are in agreement with the non renormalizability theorems of this model, were already known in perturbation theory.

5. The approximation scheme

In order to study the structure functions of the two points correlations functions, let us consider eq.s (4.10), (4.11). As we have already observed they are equivalent to an infinite system of flow equations; therefore a truncation procedure is needed. The important observation is that the correlation vertex functions, which have a different field content with respect to the original classical Navier-Stokes action, are relevant in the infrared region, i,e. when the x-like variables are near unity, but they are less relevant in the inertial region where $x \gg 1$. The physical reason for this expected behavior is due to the statistical fluctuations induced by the non local vertex $2ih(q;m)P^{\alpha\beta}(q)\hat{v}^{\alpha}(-\hat{q})\hat{v}^{\beta}(\hat{q})$ which has a very concentrated Fourier spectrum in the long wave length region. The non linear terms transfer, by the cascade mechanism, these fluctuations to all regions of the Fourier spectrum, but the speed of this transfer depends on the coupling of the non linear term with the degrees of freedom of the system; in other words on the Reynolds number. If the Reynolds number is very large the statistical fluctuations are swiftly transferred to a spectral region where the dissipative effects are dominant and, in order to preserve the stationary regime, the corresponding amplitudes are decreasing for increasing momenta. A more specific analysis follows from the connection between the parameters D_0, m, the quantities related to the regime of the physical system such as the Reynolds number, and the rate of power dissipated by a mass unit $\mathcal{E}$ of fluid. From the normalization condition (2.5) we have for the dimensionless quantity $\frac{D_0}{m^4}$

$$\frac{D_0}{m^4} = \frac{\pi^2}{\int dyy^2\chi(y)}\frac{\langle\mathcal{E}\rangle}{m^4}.$$

The Kolmogorov's cutoff is defined by $K_d^4 = \frac{\mathcal{E}}{\nu^3}$ with ν the kinematic viscosity. Recalling then (4.1) the following relation is obtained

$$\frac{D_0}{m^4} = \frac{\pi^2}{\int dyy^2\chi(y)}\nu^3\mathcal{R}^3. \tag{5.1}$$

As we see the statistical fluctuations vanish for $\mathcal{R} \to 0$ according to the fact that, in this case, no transfer of energy in a spectral region different from m is possible and the energy pumped into the system at this scale is immediately dissipated (indeed for $\mathcal{R} \to 0$ $m \gg K_d$).

If we analyze in more detail eq.s (4.10), (4.11) we find in the r.h.s. the three and four points proper vertex correlation functions

$$\Gamma_{\tilde{v}^{\rho}(\hat{q})\tilde{v}^{\alpha}(\hat{p})\hat{v}^{s}(0)}, \quad \Gamma_{\tilde{v}^{r}(\hat{q}+\hat{p})\tilde{v}^{\delta}(-\hat{q})\hat{v}^{\beta}(0)}, \quad \Gamma_{\tilde{v}^{\rho}(\hat{q})\tilde{\tilde{v}}^{\alpha}(\hat{p})\hat{v}^{s}(0)} \tag{5.2}$$

$$\Gamma_{\tilde{v}^{\rho}(\hat{q})\tilde{v}^{\delta}(-\hat{q})\tilde{v}^{\alpha}(\hat{p})\hat{v}^{\beta}(0)}, \quad \Gamma_{\tilde{v}^{\rho}(\hat{q})\tilde{v}^{\delta}(-\hat{q})\tilde{\tilde{v}}^{\alpha}(\hat{p})\hat{v}^{\beta}(0)} \tag{5.3}$$

and the two points proper correlation functions appear in the full propagators as results from (4.7). Concerning these last correlation functions we have a first observation about the function $\Gamma_{v^\alpha \hat{v}^\beta}$. From (4.16) we rewrite the function $f(\tau, x, ...)$ isolating the terms which depend on the τ variable i,e.

$$f\left(\tau, x; \frac{D_0}{m^4}\right) = f\left(x; \frac{D_0}{m^4}\right) + f_\tau\left(\tau, x; \frac{D_0}{m^4}\right).$$

From the Ward identity (3.12) and (4.17) we obtain

$$\frac{\partial}{\partial \tau} f_\tau\left(\tau, x; \frac{D_0}{m^4}\right) x^2 = i\Sigma\left(\tau, 0, x, 0; \frac{D_0}{m^4}\right) \tag{5.4}$$

i. e. $f_\tau\left(\tau, x; \frac{D_0}{m^4}\right) x^2$ has the same scaling behavior of the statistical corrections to the three points classical vertex. Therefore, for the previous argument, we expect that f_τ decreases faster then f. In a completely similar way we see that all τ dependent corrections of the vertex functions with n v and m $\hat{v}$ fields, are connected, by the Ward identities, to the functions with $n+1$ v and m $\hat{v}$ fields. Writing

$$f_\tau\left(\tau, x; \frac{D_0}{m^4}\right) x^2 = -i\tau g\left(x; \frac{D_0}{m^4}\right) + \tau^2 g_{(2)}\left(x; \frac{D_0}{m^4}\right) + ... + \tau^n g_{(n)}\left(x; \frac{D_0}{m^4}\right) + ..., \tag{5.5}$$

from the last observations we have that $g\left(x; \frac{D_0}{m^4}\right)$ is related to three point vertex statistical corrections, $g_{(2)}\left(x; \frac{D_0}{m^4}\right)$ is related to the four point vertex and so on. Taking into account the previous considerations we can write the flow equation for $f(x; \frac{D_0}{m^4})$, where we analyze only the leading contributions appearing in the triple vertex. In other words considering the vertex functions that appear in the r. h. s. of equation (4.10), i. e. the three and four points function $\Gamma^m_{\tilde{v}^\lambda(\eta,y)\tilde{v}^\alpha(-\eta,x-y)\hat{v}^\beta(0)}$, $\Gamma^m_{\tilde{v}^\rho(\eta,y)\tilde{v}^\delta(-\eta,-y)\tilde{v}^\alpha(0,x)\hat{v}^\beta(0)}$, we separate the leading contribution from the statistical correction by writing the vertex $\Gamma^m_{vv\hat{v}}$ as

$$\Gamma^m_{vv\hat{v}} = \Gamma^{(0)}_{vv\hat{v}} + (\Delta\Gamma)_{vv\hat{v}} \tag{5.6}$$

and we suppose, for $x \gg 1$, the following approximations

$$\lim_{x\gg 1} \Gamma^m_{vv\hat{v}} \sim \Gamma^{(0)}_{vv\hat{v}} \quad and \quad \lim_{x\gg 1} \Gamma^m_{vvv\hat{v}} \sim 0. \tag{5.7}$$

The same analysis is performed for eq.(4.11) where the vertices involved are $\Gamma^m_{\tilde{v}^\lambda(\eta,y)\tilde{v}^\rho(-\eta,x-y)\hat{v}^\beta(0)}$, $\Gamma^m_{\tilde{v}^\lambda(\eta,y)\tilde{\hat{v}}^\rho(-\eta,x-y)\hat{v}^\beta(0)}$, $\Gamma^m_{\tilde{v}^\lambda(\eta,y)\tilde{v}^\rho(-\eta,-y)\tilde{\hat{v}}^\alpha(0,x)\hat{v}^\beta(0)}$, with the approximations

$$\lim_{x\gg 1} \Gamma^m_{v\hat{v}\hat{v}} \sim 0 \quad and \quad \lim_{x\gg 1} \Gamma^m_{vv\hat{v}\hat{v}} \sim 0. \tag{5.8}$$

In this way eq.s (4.10), (4.11) give a closed system of differential equations. The approximation here proposed can also be obtained by a suitable field truncation, preserving the Galilei-invariance, in the effective irreducible action [19].

In order to write, from (4.10) and (4.11), the equations for the structure functions $f\left(x;\frac{D_0}{m^4}\right)$ and $M\left(x;\frac{D_0}{m^4}\right)$ we define

$$f\left(x;\frac{D_0}{m^4}\right) \equiv \left(\frac{D_0}{m^4}\right)^{\frac{1}{3}} \phi_m(x), \tag{5.9}$$

$$M\left(x;\frac{D_0}{m^4}\right) \equiv M_m(x). \tag{5.10}$$

and after straightforward calculations we obtain

$$\begin{aligned} &\left(m\partial_m - x\partial_x - \tfrac{4}{3}\right)\phi_m(x) \\ &= \int \frac{y^2 dy}{4\pi^2} \int_{-1}^{1} dt \frac{(1-t^2)\left(t\left(\frac{y}{x}-2\frac{x}{y}\right)-\frac{x^2}{y^2}\right)(3+y\partial_y)\chi(y)}{\phi_m(y)(x^2+y^2+2txy)\left[\phi_m\left(\sqrt{x^2+y^2+2txy}\right)(x^2+y^2+2txy)+\phi_m(y)y^2\right]}, \end{aligned} \tag{5.11}$$

$$\begin{aligned} &\left(m\partial_m - x\partial_x - 3\right)M_m(x) \\ &= \int \frac{y^2 dy}{4\pi^2} \int_{-1}^{1} dt \frac{(1-t^2)x^2\left(1+2t^2+\frac{x^2}{y^2}+3t\frac{x}{y}\right)(3+y\partial_y)\chi(y)\left(\chi_m\left(\sqrt{x^2+y^2+2txy}\right)+M_m\left(\sqrt{x^2+y^2+2txy}\right)\right)}{\phi_m(y)\phi_m\left(\sqrt{x^2+y^2+2txy}\right)(x^2+y^2+2txy)^2\left[\phi_m\left(\sqrt{x^2+y^2+2txy}\right)(x^2+y^2+2txy)+\phi_m(y)y^2\right]}. \end{aligned} \tag{5.12}$$

Concerning the boundary and initial conditions for (5.11) and (5.12) we must consider that these equations are valid only in the inertial region, i.e. for $x \gg 1$, provided $x \ll \frac{K_d}{m} \equiv x_d$. As the parameter m is very small compared with K_d (we are studying the system near the infinite volume limit) it follows that $x_d \sim \infty$ is a good approximation. Recalling that the Kolmogorov scale K_d is related to the viscosity and to the rate of energy dissipation by the relation $K_d \approx \left(\frac{\mathcal{E}}{\nu^3}\right)^{\frac{1}{4}}$ [7], the previous approximation, for $m \neq 0$, is equivalent to $\nu \sim 0$. In conclusion we extrapolate the following boundary conditions:

$$\phi_m(\infty) = M_m(\infty) = 0, \quad for \quad m \neq 0. \tag{5.13}$$

It is clear that (5.11) and (5.12) with (5.13) enable us to describe the physical system in the dissipative region. Considering that we are studying the system near the supposed fixed point in the infinite volume limit, the initial conditions are given for $m \to 0$. As we have already observed our model corresponds to a regularized theory and, for vanishing regularization parameter m, the infrared divergences, due to the infinite volume limit, will appear again. The choices (5.9),

(5.10) fix these divergences according to the power counting and dimensional analysis; in $\phi_m(x)$ and $M_m(x)$ no explicit dependence on the parameter m is present. It is also evident that for $x \sim 1$ the considered equations are inadequate for the description of the physical system.

The meaning of our approximation scheme is evident by observing eq.s(5.11), (5.12) which correspond to the first step of the loop expansion. However these equations contain a nonperturbative contribution concerning the structure functions (5.9) and (5.10) which are solutions of the same equations with the initial conditions for $m \to 0$.

The next step requires the one loop calculation of the vertex functions $(\Delta\Gamma)_{vv\hat{v}}$, $\Gamma^m_{vvv\hat{v}}$, $\Gamma^m_{v\hat{v}\hat{v}}$ and $\Gamma^m_{vv\hat{v}\hat{v}}$ still using the ERG containing in the r. h. s. only the leading terms and the structure functions (5.9) and (5.10). The two loops contributions follow inserting these vertices in the original equations (4.10), (4.11). This procedure is extendible to an arbitrary order and it contains a non perturbative input given by the structure functions (5.9) and (5.10). Naturally this scheme is meaningful only if the two loop are smaller then the one loop contributions, and so on. We also observe that, in our approach, no particular hypothesis concerning the effective action is needed and this result follows from a self-consistency argument of our model.

6. Numerical results

In this section we report some numerical results concerning the solutions of the system (5.11), (5.12). The analysis takes into account different choices of the stirring force, all satisfying the spectral condition discussed in the previous sections. By the expected universality in the inertial range we must verify that the exact shape of the noise correlator is not very important. We consider a $\chi(x)$ function with the general form

$$\chi(x) = x^s e^{-x^{2n}} \tag{6.1}$$

where s and n are positive integers. Eq.s (5.11), (5.12) are solved with an explicit finite-difference scheme starting at the boundary conditions (5.13) and by the initial conditions which require that no explicit m-dependence is present for $m \to 0$ (i. e. for the system near the critical point). The numerical computations show that the solutions maintain explicit independence on the scale-parameter m. The function $\phi(x)$ and $M(x)$ are computed in the range $10^{-1} < x < 10^2$. The inertial range corresponds to $x \gg 1$ and in this region the function $\phi(x)$ and $M(x)$ reach a scale invariant regime. Indeed, referring to the parametrization

$$\phi(x) = \sigma x^{-\frac{4}{3}+\alpha} \quad and \quad M(x) = \gamma x^{-3+\beta}, \tag{6.2}$$

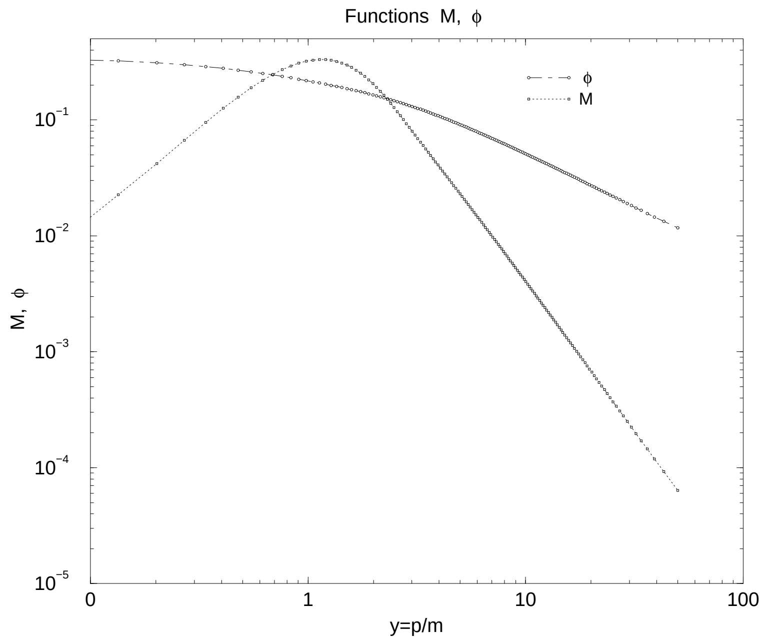

Figure 1: ϕ and M functions

the numerical computation gives for α and β a value $\sim +0.37$ in the region $x > 1$ and the function $\phi(x)$ and $M(x)$ are graphically represented in Fig.(1). The detailed numerical values of the parameters related to the functions $\phi(x)$ and $M(x)$ obtained with different noise functions χ are summarized in table (1) and the substantial independence of the exponents $-\frac{4}{3}+\alpha$ and $-3+\beta$ on the particular form of $\chi(x)$ appears very clearly.

$\chi(x)$	σ (± 0.001)	$-\frac{4}{3}+\alpha(\pm 0.0005)$	γ (± 0.001)	$-3+\beta$ (± 0.0005)	$\frac{D_0}{m^4}\,(\nu\mathcal{R})^{-3}$
$x^2 e^{-x^2}$	0.899	-0.924	20.00	-2.590	14.85
$x^2 e^{-x^4}$	0.900	-0.935	22.49	-2.601	43.55
$x^2 e^{-x^6}$	0.935	-0.930	26.02	-2.630	52.46
$x^4 e^{-x^2}$	0.924	-0.933	29.63	-2.599	5.94
$x^4 e^{-x^4}$	0.994	-0.929	28.97	-2.595	42.95
$x^6 e^{-x^6}$	1.02	-0.930	33.55	-2.596	66.81
$x^8 e^{-x^8}$	1.03	-0.930	35.93	-2.633	88.82
$x^{10} e^{-x^{10}}$	1.08	-0.929	38.05	-2.595	109.97
$x^{12} e^{-x^{12}}$	1.09	-0.931	38.41	-2.597	109.97
$e^{-\frac{1}{x^2}} e^{-x^2}$	0.942	-0.922	27.70	-2.588	54.86

Table 1: Parameter of the functions $\phi(x)$ and $M(x)$ for various stirring forces.

A significative check of our analysis is given by the computation of the energy spectrum. From

$$E(\hat{x};m) = -\frac{1}{2}\left.\frac{\delta^2\mathcal{Z}}{\delta\hat{J}^\alpha(\hat{x})\delta\hat{J}_\alpha(\hat{x})}\right|_{\mathcal{J}=0} = \frac{1}{2}\langle u^\alpha(t,\vec{x})u_\alpha(t,\vec{x})\rangle_c$$

$$= \frac{1}{2(2\pi)^4}\int dq^0 q^2 dq d\Omega_q \frac{1}{\Gamma^m_{\tilde{u}^\alpha\hat{u}^\gamma}(\hat{q})}\Gamma^m_{\tilde{\tilde{u}}^\gamma\hat{u}^\rho}(\hat{q})\frac{1}{\Gamma^m_{\tilde{\tilde{u}}^\rho u^\alpha}(\hat{q})} = \int dq E(q;m) \tag{6.3}$$

we obtain

$$E(q;m) = \frac{1}{2\pi^2}q^2\frac{M(\frac{q}{m})}{\phi(\frac{q}{m})q^2}m^{-\frac{5}{3}}D_0^{\frac{2}{3}} = \frac{1}{2}\left(\frac{\gamma}{\sigma}\right)\left(\frac{1}{\int dyy^2\chi(y)}\right)^{\frac{2}{3}}\mathcal{E}^{\frac{2}{3}}q^{-\frac{5}{3}}\left(\frac{q}{m}\right)^{\beta-\alpha}. \tag{6.4}$$

Recalling (4.2) the Kolmogorov constant C_K is given by the q-independent part of (6.4) i.e.

$$C_K = \frac{1}{2}\left(\frac{\gamma}{\sigma}\right)\left(\frac{1}{\int dyy^2\chi(y)}\right)^{\frac{2}{3}}. \tag{6.5}$$

The energy power behavior in terms of the q-variable is given by the exponent $\xi = -\frac{5}{3}+\beta-\alpha$. The numerical values of C_K and ξ, computed with different noise functions $\chi(x)$ are summarized in the following table

$\chi(x)$	ξ (± 0.001)	$C_K(\pm 0.002)$	$\chi(x)$	ξ (± 0.001)	$C_K(\pm 0.002)$
$x^2e^{-x^2}$	-1.666	1.124	$x^4e^{-x^4}$	-1.666	1.461
$x^2e^{-x^4}$	-1.667	1.267	$x^6e^{-x^6}$	-1.666	1.660
$x^2e^{-x^6}$	-1.666	1.417	$x^8e^{-x^8}$	-1.666	1.767
$e^{-\frac{1}{x^2}}e^{-x^2}$	-1.667	1.489	$x^{10}e^{-x^{10}}$	-1.666	1.784
$x^4e^{-x^2}$	-1.667	1.624	$x^{12}e^{-x^{12}}$	-1.666	1.785

Table 2: Slope ξ and Kolmogorov constant C_K for various stirring forces

Looking at table (2) the exponent ξ in (6.4) appears to be practically independent on the particular form of $\chi(x)$ and corresponds to the Kolmogorov's value $-\frac{5}{3}$. The constant C_K is stable for very narrow χ functions and approaches the value $C_K \approx 1.78$ compatible with the experimental results [20]. These results are summarized, for the energy spectrum $E(q;m)$, in Fig.(2).

We conclude this section giving a numerical justification of the loop expansion here considered. A rather rough evaluation in Appendix B shows that the two and one loop terms

$$\int \frac{d\hat{q}}{(2\pi)^4}m\partial_m h(q;m)\Delta_{\tilde{v}^\gamma v^\rho}(\hat{q};m)\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\delta(-\hat{q})\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}\Delta_{v^\delta\tilde{v}^\gamma}(\hat{q};m), \tag{6.6}$$

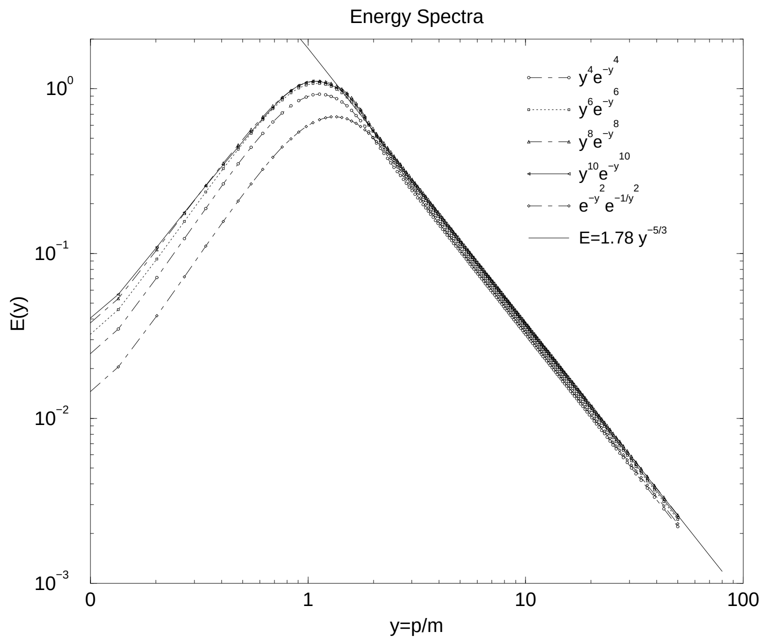

Figure 2: Energy spectra for various $\chi(x)$ functions

$$\int \frac{d\hat{q}}{(2\pi)^4} m\partial_m h(q;m)\Delta_{\hat{v}^\gamma v^\rho}(\hat{q};m)\Gamma^{(0)}_{\tilde{v}^\rho(\hat{q})\tilde{v}^\alpha(\hat{p})\hat{v}^s(0)}\Delta_{\hat{v}^s v^r}(\hat{q}+\hat{p})\Gamma^{(0)}_{\tilde{v}^r(\hat{q}+\hat{p})\tilde{v}^\delta(-\hat{q})\hat{v}^\beta(0)}\Delta_{v^\delta \hat{v}^\gamma}(\hat{q};m) \quad (6.7)$$

are in the ratio $\frac{1}{n}$ i.e. proportional to the spectrum size of the noise function $h(x;m)$. Analogous results are obtained for the other two loops terms and in general the terms of our loops ordering, generated by the ERG, are decreasing as inverse powers of n. This result is not a sufficient argument for the validity of the considered loops expansion, but only a proof of consistency of the theory and a precise numerical estimation of all interesting quantities is very hard. We also observe that, unlike a naive perturbative loops expansion, the terms of the expansion are not increasing as a power of the Reynolds number $\mathcal{R}$.

7. Conclusions

The numerical results in the previous section are encouraging and, as reported in appendix B, an essential ingredient appears to be the size of the spectra of two point stirring force correlation function. If the approximation scheme here proposed is physically meaningful, the non-perturbative contributions are mainly due, in the inertial region, to the two point function

$\Gamma^m_{u\hat{u}}$ and, particularly, to the anomalous dimension of the $\hat{u}$ field (no anomalous dimension is associated to the u field as results by the Ward Identity). In other words, near a critical point, the ERG provides a summability criterion concerning the two point function and the residual non classical dynamics is described by perturbative corrections. We cannot provide a proof for this scenario, but only support it with plausibility arguments since a direct check requires a very heavy numerical analysis. We think that an interesting alternative to the numerical analysis is given by the method recently proposed by T. R. Morris [13], [14] in the context of quantum field theory. The Morris's method is similar to our analysis in the treatment of the field anomalous dimensions in the ERG, but our perturbative loops expansion is replaced by a more promising nonperturbative approximation of the effective action. The Morris's scheme correspond to a quasi-local approximation of the effective action obtained by a derivative expansion. However the application of this method to our model is not straightforward, in particular Morris approach does not preserve Galilei invariance. It should however be interesting to compare numerical results obtained with different methods in the effort towards a better understanding of the different approximations in the description of the physics of these systems.

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Appendix A

In this appendix we prove that the vertex correlation functions containing only v^α fields necessarily vanish for an arbitrary value of the parameter $m \neq 0$ if these function vanish for a given $m = \bar{m}$. Consider the flow equation for any such function; from (4.6) we have

$$m\partial_m \Gamma^m_{\tilde{v}^{\alpha_1}(\hat{p}_1)\dots v^{\alpha_n}(0)} = -\frac{i}{2}\int \frac{d\hat{q}}{(2\pi)^4} m\partial_m R_{ab}(q;m)\Delta_{\Psi_b\Psi_c}(q;m)\; \frac{\delta^{2+n}\bar{\Gamma}(\Psi;m)}{\delta\tilde{\Psi}_c(\hat{q})\delta\tilde{\Psi}_l(-\hat{q})\delta\tilde{v}^{\alpha_1}(\hat{p}_1)\dots\delta v^{\alpha_n}(0)}\Big|_{\Psi=0} \Delta_{\Psi_l\Psi_a}(q;m). \tag{A.1}$$

It is straightforward to see that all terms in the r. h. s. of eq.(A.1) contain at least one vertex with only the v^α field or the component $\Delta_{v^\alpha v^\beta}(q;m)$ of the full propagator. For this reason

$$\partial_m \Gamma^m_{\tilde{v}^{\alpha_1}(\hat{p}_1)\dots v^{\alpha_n}(0)}\Big|_{m=\bar{m}} = 0$$

and at the generic point $\bar{m}+\delta m$ near $\bar{m}$ we have

$$\Gamma^{\bar{m}+\delta m}_{\tilde{v}^{\alpha_1}(\hat{p}_1)\dots v^{\alpha_n}(0)} = O\left((\delta m)^2\right) \tag{A.2}$$

which iteratively yields the desired result. In other words we have a non-renormalization property for these vertices and they will be absent if they are not present in the original *classical* action.

Appendix B

Here we propose a heuristic argument in order to compare two loops with one loop terms like the ones considered at the end of section 6. Taking into account the results of section 6, consider the two point functions

$$\Gamma_{\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)} \to \Gamma^m_{\tilde{v}^\alpha(\tau,x)\hat{v}^\beta(0)} \sim m^2\left(-i\tau+\sigma\left(\frac{D_0}{m^4}\right)^{\frac{1}{3}}x^{1+\omega}\right)P^{\alpha\beta}(n_p). \tag{B.1}$$

and

$$\Gamma_{\tilde{\tilde{v}}^\alpha(\hat{p})\hat{v}^\beta(0)} \to \Gamma^m_{\tilde{\tilde{v}}^\alpha(\tau,x)\hat{v}^\beta(0)} \sim 2i\frac{D_0}{m^3}\left(\chi(x)+\gamma x^{-3+\beta}\right)P^{\alpha\beta}(n_p) \tag{B.2}$$

where, from Table (1), $\omega = 1+\left(-\frac{4}{3}+\alpha\right) \sim 0.07$ and $\beta \sim 0.4$. The calculation of the vertex function $\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\delta(-\hat{q})\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}$ is performed by the ERG where, as in the case of the two point function, only the leading vertex is taken into account. In this approximation the ERG which we consider is

$$\begin{aligned}
&m\partial_m\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\delta(-\hat{q})\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}\\
&= -2\int\frac{d\hat{k}}{(2\pi)^4}m\partial_m h(k;m)\Delta_{\hat{u}^f u^b}(\hat{k};m)\left\{\Gamma^{(0)}_{u^b(\hat{k})\hat{u}^\beta(-\hat{p})u^s(\hat{k}-\hat{p})}\Delta_{u^s\hat{u}^l}(\hat{k}-\hat{p};m)\right.\\
&\left(\Gamma^{(0)}_{\hat{u}^l(\hat{k}-\hat{p})u^\delta(-\hat{q})u^m(-\hat{k}+\hat{p}+\hat{q})}\Delta_{u^m\hat{u}^n}(\hat{k}-\hat{p}-\hat{q};m)\Gamma^{(0)}_{\hat{u}^n(\hat{k}-\hat{p}-\hat{q})u^\rho(\hat{q})u^a(-\hat{k}+\hat{p})}\right.\\
&\left.+\Gamma^{(0)}_{\hat{u}^l(\hat{k}-\hat{p})u^\rho(\hat{q})u^m(-\hat{k}+\hat{p}-\hat{q})}\Delta_{u^m\hat{u}^n}(\hat{k}-\hat{p}+\hat{q};m)\Gamma^{(0)}_{\hat{u}^n(\hat{k}-\hat{p}+\hat{q})u^\delta(-\hat{q})u^a(-\hat{k}+\hat{p})}\right)\\
&\Delta_{u^a\hat{u}^d}(\hat{k}-\hat{p};m)\Gamma^{(0)}_{\hat{u}^d(\hat{k}-\hat{p})u^\alpha(\hat{p})u^c(-\hat{k})}+\left[\Gamma^{(0)}_{u^b(\hat{k})\hat{u}^\beta(-\hat{p})u^s(\hat{k}-\hat{p})}\Delta_{u^s\hat{u}^l}(\hat{k}-\hat{p};m)\right.\\
&\left(\Gamma^{(0)}_{\hat{u}^l(\hat{k}-\hat{p})u^\delta(-\hat{q})u^m(-\hat{k}+\hat{p}+\hat{q})}\Delta_{u^m\hat{u}^n}(\hat{k}-\hat{p}-\hat{q};m)\Gamma^{(0)}_{\hat{u}^n(\hat{k}-\hat{p}-\hat{q})u^\alpha(\hat{p})u^a(-\hat{k}+\hat{q})}\right.\\
&\Delta_{u^a\hat{u}^d}(\hat{k}-\hat{q};m)\Gamma^{(0)}_{\hat{u}^d(\hat{k}-\hat{q})u^\rho(\hat{q})u^c(-\hat{k})}+\Gamma^{(0)}_{\hat{u}^l(\hat{k}-\hat{p}-\hat{q})u^\rho(\hat{q})u^m(-\hat{k}+\hat{p})}\Delta_{u^m\hat{u}^n}(\hat{k}-\hat{p}+\hat{q};m)
\end{aligned}$$

$$\Gamma^{(0)}_{\hat{u}^n(\hat{k}-\hat{p}+\hat{q})u^\alpha(\hat{p})u^a(-\hat{k}-\hat{q})}\Delta_{u^a\hat{u}^d}(\hat{k}+\hat{q};m)\Gamma^{(0)}_{\hat{u}^d(\hat{k}+\hat{q})u^\delta(-\hat{q})u^c(-\hat{k})}\Big)+\alpha\leftrightarrow\beta\Big]$$
$$+\Big[\Gamma^{(0)}_{u^b(\hat{k})\hat{u}^\beta(-\hat{p})u^s(\hat{k}-\hat{p})}\Delta_{u^s\hat{u}^l}(\hat{k}-\hat{p};m)\Gamma^{(0)}_{\hat{u}^l(\hat{k}-\hat{p})u^\alpha(\hat{p})u^m(-\hat{k})}\Delta_{u^m\hat{u}^n}(\hat{k};m)$$
$$\Big(\Gamma^{(0)}_{\hat{u}^n(\hat{k})u^\rho(\hat{q})u^a(-\hat{k}-\hat{q})}\Delta_{u^a\hat{u}^d}(\hat{k}+\hat{q};m)\Gamma^{(0)}_{\hat{u}^d(\hat{k}+\hat{q})u^\delta(-\hat{q})u^c(-\hat{k})}+$$
$$\Big(\Gamma^{(0)}_{\hat{u}^n(\hat{k})u^\delta(-\hat{q})u^a(-\hat{k}+\hat{q})}\Delta_{u^a\hat{u}^d}(\hat{k}-\hat{q};m)\Gamma^{(0)}_{\hat{u}^d(\hat{k}-\hat{q})u^\rho(\hat{q})u^c(-\hat{k})}\Big)+\alpha\leftrightarrow\beta\Big]$$
$$+\Big(\Gamma^{(0)}_{u^b(\hat{k})u^\delta(-\hat{q})\hat{u}^s(\hat{k}-\hat{q})}\Delta_{\hat{u}^s u^l}(\hat{k}-\hat{q};m)\Gamma^{(0)}_{u^l(\hat{k}-\hat{q})\hat{u}^\beta(-\hat{p})u^m(-\hat{k}+\hat{q}+\hat{p})}\Delta_{u^m\hat{u}^n}(\hat{k}-\hat{q}-\hat{p};m)$$
$$\Gamma^{(0)}_{\hat{u}^n(\hat{k}-\hat{q}-\hat{p})u^\alpha(\hat{p})u^a(-\hat{k}+\hat{q})}\Delta_{u^a\hat{u}^d}(\hat{k}-\hat{q};m)\Gamma^{(0)}_{\hat{u}^d(\hat{k}-\hat{q})u^\rho(\hat{q})u^c(-\hat{k})}+\delta\leftrightarrow\rho\Big)\Big\}\,\Delta_{u^c\hat{u}^f}(\hat{k};m). \quad \text{(B.3)}$$

We rescale the variables $(0,p^\alpha)$, (q^0,q^ρ) and (k^0,k^δ) as

$$p^\alpha=mxn^\alpha,\quad q^0=m^2z^0,\quad q^\rho=mzn^\rho,\quad k^0=m^2y^0,\quad k^\delta=myn^\delta \quad \text{(B.4)}$$

where n^α, n^ρ and n^δ are unitary vectors along the directions of p^α, q^ρ and k^δ. We also observe that the value of $\left|\vec{k}\right|$ is constrained, by the support of the function $h(k;m)$ (see (4.5) and (6.1)), into a narrow region centered around $k\sim m$, so that, from (B.4), $y\sim 1$. Moreover, in order to evaluate the two loops term (6.6), the variable z has also the same constraint. From the last observation it follows that, in the inertial region (where $x\gg 1$), we can consider $\frac{y}{x}\sim\frac{z}{x}\ll 1$. Taking into account only the leading terms we obtain from eq.(B.3), after tedious but straightforward calculations

$$\left(m\partial_m-x\partial_x-z\partial_z-2z^0\partial_{z^0}\right)\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\delta(-\hat{q})\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}$$
$$\sim\frac{1}{6\pi^2}\frac{1+\omega}{n}\Gamma\left(\frac{2+s-\omega}{2n}\right)\left(\frac{D_0}{m^4}\right)^{\frac{1}{3}}\left[\frac{x^{1-\omega}n_p^s n_p^\rho P^{\alpha\beta}(n_p)}{\sigma^2\left((z^0)^2+\sigma^2\left(\frac{D_0}{m^4}\right)^{\frac{2}{3}}x^{2+2\omega}\right)}+\ldots\right]+\int dy\chi(y)O\left(\frac{y}{x},\frac{z}{x}\right). \quad \text{(B.5)}$$

Where we have considered $\chi(y)=y^se^{-y^{2n}}$ as in (6.1). A solution of eq.(B.5) is not obtainable with elementary procedures and requires numerical methods, but it is still possible to extract sufficient information concerning our purpose with very simple arguments. We simplify (B.5) rescaling the variable z^0 as

$$z^0=\sigma\left(\frac{D_0}{m^4}\right)^{\frac{1}{3}}x^{1+\omega}w^0 \quad \text{(B.6)}$$

and setting

$$\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\delta(-\hat{q})\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}=\frac{1}{6\pi^2}\frac{1+\omega}{n}\Gamma\left(\frac{2+s-\omega}{2n}\right)\left(\frac{D_0}{m^4}\right)^{-\frac{1}{3}}\frac{x^{-(1+3\omega)}}{\sigma^4}G(m,x,w^0)n_p^\delta n_p^\rho P^{\alpha\beta}(n_p)+\ldots \quad \text{(B.7)}$$

The function $G(m,x,w^0)$ satisfies the equation

$$\left(m\partial_m - x\partial_x - 2w^0\partial_{w^0}\right)G(m,x,w^0) - \left(\frac{1}{3} - 3\omega\right)G(m,x,w^0) = \frac{1}{1+(w^0)^2}.$$

Inserting (B.7) in (6.6) we obtain

$$\begin{aligned}&\int \tfrac{d\hat{q}}{(2\pi)^4} m\partial_m h(q;m)\Delta_{\hat{v}^\gamma v^\rho}(\hat{q};m)\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\delta(-\hat{q})\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}\Delta_{v^\delta\hat{v}^\gamma}(\hat{q};m)\\ &= \tfrac{m^2}{12\pi^4}\tfrac{1+\omega}{n}\Gamma\left(\tfrac{2+s-\omega}{2n}\right)\left(\tfrac{D_0}{m^4}\right)^{\frac{1}{3}}\tfrac{x^{-2\omega}}{\sigma^5}P^{\alpha\beta}(n_p)\int dz z^{-2\omega}\left(3+s-2nz^{2n}\right)z^s e^{-z^{2n}}G\left(m,x,\tfrac{z}{x}\right).\\ &+............\end{aligned}$$

Considering the leading term $\left(G\left(m,x,\frac{z}{x}\right) = G'\left(m,x\right) + O\left(\frac{z}{x}\right)\right)$ we have

$$\begin{aligned}&\int \tfrac{d\hat{q}}{(2\pi)^4} m\partial_m h(q;m)\Delta_{\hat{v}^\gamma v^\rho}(\hat{q};m)\Gamma_{\tilde{v}^\rho(\hat{q})\tilde{v}^\delta(-\hat{q})\tilde{v}^\alpha(\hat{p})\hat{v}^\beta(0)}\Delta_{v^\delta\hat{v}^\gamma}(\hat{q};m)\\ &= \tfrac{m^2}{12\pi^4}\left(\tfrac{1+\omega}{n}\right)^2\Gamma\left(\tfrac{2+s-\omega}{2n}\right)\Gamma\left(\tfrac{1+s-2\omega}{2n}\right)\left(\tfrac{D_0}{m^4}\right)^{\frac{1}{3}}\tfrac{x^{-2\omega}}{\sigma^5}P^{\alpha\beta}(n_p)G'\left(m,x\right)+ ....\end{aligned}$$

The Euler's functions $\Gamma\left(\frac{2+s-\omega}{2n}\right)$ and $\Gamma\left(\frac{1+s-2\omega}{2n}\right)$ are of order of unity. With the same method we prove that (6.7) is order $O\left(\frac{1}{n}\right)$. The previous arguments show a decreasing with the powers of the spectrum size of the noise function $h(x;m)$ for the considered loops ordering. This argument gives a meaning to the last term of Tab:2 in Section 6. Finally we observe that the previous discussion is not a proof, but only a consistency argument with the hypothesis of Section 5.

References

[1] D. Foster, D. Stephen, M. Nelson, Large distance and long time properties of a randomly stirred fluid, Phys. Rev.**A 16** (1977) 732

[2] C. DeDominicis, P. C. Martin, Energy Spectra of Certain Randomly-Stirred Fluids, Phys. Rev.A**19** (1979) 419

[3] V. Yakhot, S. A. Orzag, Renormalization Group Analysis of Turbulence, I. Basic theory, J. Sci. Comp. **1** (1986) 3

[4] A. N. Kolmogorov, C.R. Acad. Sci. URSS **30** (1941) 301 - **32** (1941) 16

[5] G. Eyink, The Renormalization Group in Statistical Hydrodynamics, Phys. Fluids **6** (1994) 3063

[6] G. Eyink, turbulence Noise, Jour. of Stat. Phys. **83** (1996) 955

[7] W. D. McComb, The Physics of Fluid turbulence, Claredon Press. Oxford (1994)

[8] J. Polchinski, Nucl. Phys. **B231** (1984) 269

[9] G. Gallavotti, Rev. Mod. Phys. **57** (1985) 471

[10] C. Becchi, On the construction of renormalized quantum field theory using renormalization group techniques, Lectures notes of the 1991 Italian Doctoral school in Parma

[11] P. C. Martin, E. D. Siggia, H. A. Rose, Statistical Dynamics of Classical System, Phys. Rev. **A8** (1973) 423

[12] M. Bonini, M. D'Attanasio, G. Marchesini, Nucl. Phys. **B418** (1994) 81

[13] T. R. Morris, Phys. Lett. **B329** (1994) 241, **334** (1994) 355, Nucl. Phys. **42**, Proc. Suppl. (1995) 811

[14] T. R. Morris, Momentum scale expansion of sharp cutoff flow equation, Nucl. Phys.**B458** (1996) 477

[15] L. Landau, E. M. Lifshitz, Physique théorique, Tome 6, Éditions MIR Moscou (1989).

[16] U. Frisch, turbulence, Cambridge University Press (1995)

[17] P. Brax, A Renormalization group study of the three dimensional turbulence, preprint, hep-th/9607230

[18] C. Wetterich, Nucl. Phys. **B352** (1991) 529 D. Ronis, Field-theoretic renormalization group and turbulence, Phys. Rev. **A36** (1987) 3322

[19] P. Tomassini, An Exact Renormalization Group analysis of 3-d Well Developed turbulence, Phys. Lett.**B411** (1997) 117

[20] A. Praskovsky, S. Oncley, Phys. Fluids **6** (1994) 2886